Lecture 1 (1/3/22).

- We start this course in Conway, Ch VI.
- Material covered in Conway Ch I-V are assumed familiar to students.

Recall. In 220A, we proved

Max Mod Thm - I. If f is analytic in a region (open connected subset) $G \subseteq \mathbb{C}$ and $\exists\, a \in G$ s.t. $|f(z)| \leq |f(a)|$, $\forall z \in G$, then f is constant.

Pf. Proof can be given using Cauchy's Integral formula or Open Mapping Thm. Pf using OMT is simplest but the proof using CIF generalizes to harmonic functions. Details are DIY. $\blacksquare$

A simple consequence is

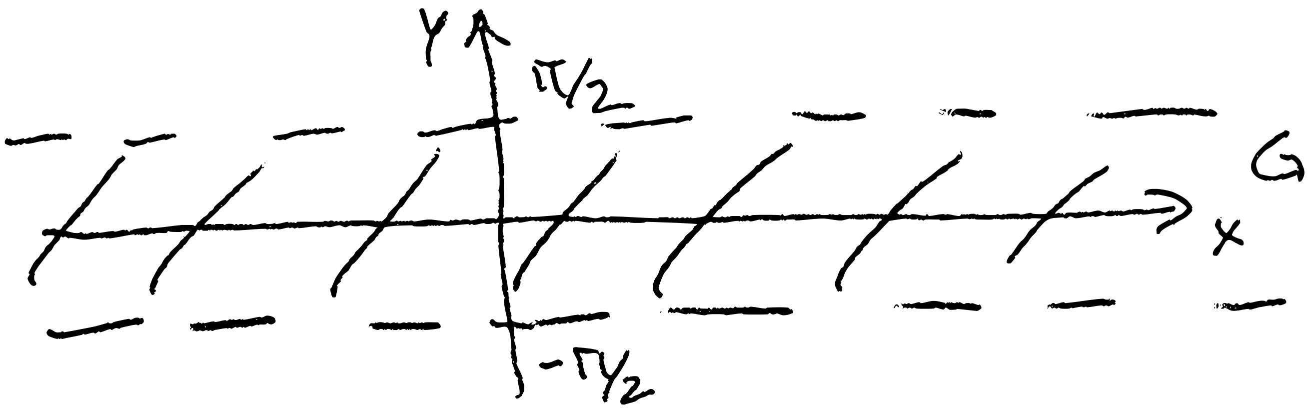
MMT-II. If G is open and bounded, f analytic in G and continuous in $\bar{G}$, then

$$\max_{z \in \bar{G}} |f(z)| = \max_{\xi \in \partial G} |f(\xi)|.$$

Note that G bdd is important in MMT-II.

Ex ① Consider $f(z) = e^{e^z}$ in

$G = \{z : |\text{Im}\, z| < \pi/2\}$.



$\Rightarrow$

$\partial G = \{\text{Im}\, z = \pi/2\} \cup \{\text{Im}\, z = -\pi/2\}$.

Let $z = x+iy$. Then $|f(z)| = e^{\text{Re}\, e^z} =$

$= e^{e^x \cos y}$. Thus, $|f(z)| = 1$ on ∂G.

However,

$|f(x)| = f(x) = e^{e^x} \to \infty$ as $x \to \infty$.

Def. ① If $G \subseteq \mathbb{C}$, then $\partial_\infty G :=$ boundary of $G \subseteq \mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$. Thus, $\partial_\infty G = \partial G$ if G is bounded and $\partial_\infty G = \partial G \cup \{\infty\}$ if G unbounded.

($\partial_\infty G \subseteq \mathbb{C}_\infty$)

② If $G \subseteq \mathbb{C}$ and $\xi \in \partial_\infty G$, then

$$\limsup_{z \to \xi} u(z) := \lim_{r \to 0^+} \sup_{z \in G \cap B(\xi, r)} u(z)$$

ball in $\mathbb{C}_\infty$ if $\xi = \infty$.

for $u : G \to \mathbb{R}$. As in calculus,

$\lim_{z \to \xi} u(z)$ exists $\iff$ $\limsup_{z \to \xi} u(z)$ $=$ $\liminf_{z \to \xi} u(z)$.

<u>Max Mod Thm - III</u>. Let f be analytic in G and assume $\limsup_{z \to \xi} |f| \le M$, $\forall \xi \in \partial_\infty G$.

Then $|f(z)| \le M$ in G.

<u>Pf.</u> Consider G as open subset of metric space $\mathbb{C}_\infty$ and $u(z) = |f(z)|$ cont. fcn $G \to \mathbb{R}$. For a contradiction, assume $\exists z \in G$ s.t. $u(z) > M$. Then $\exists \varepsilon > 0$ s.t. $G_\varepsilon = \{z \in G : u(z) > M + \varepsilon\}$ is open and nonempty. Since $\limsup_{z \to \xi} u(z) \le M$, $\forall \xi \in \partial_\infty G$, there is an open set $U_\varepsilon \subseteq \mathbb{C}_\infty$ s.t. $u(z) < M + \varepsilon$ in $U_\varepsilon \cap G$ and $\partial_\infty G \subseteq U_\varepsilon$. It follows that $\overline{G_\varepsilon}$ is compact in G. (Why? DIY.)

Going back to $\mathbb{C}$, we then have

$G_\varepsilon \subseteq G$, $\overline{G_\varepsilon}$ compact in $G \subseteq \mathbb{C}$ and hence G_ε is <u>bounded</u>. Also, f is analytic in G_ε and <u>cont.</u> in $\overline{G_\varepsilon}$. By MMT-II,

$$\max_{\overline{G_\varepsilon}} |f| = \max_{\partial G_\varepsilon} |f|.$$

But, by def., $|f| = M+\varepsilon$ on ∂G_ε and $|f(z)| > M+\varepsilon$ in G_ε $\Rightarrow G_\varepsilon = \emptyset$ which is a contradiction. $\boxtimes$